Mathematical Methods

VCE Sample Practice Examination 1

Reading time: 15 minutes
Writing time: 1 hour

STUDENT NUMBER

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Students are not permitted to bring any technology (calculator, software, mobile phones, or other unauthorized electronic devices) or notes of any kind into the examination room.

Instructions to students:

- Answer all questions in the spaces provided.
- Write your answers in English.
- In all questions where a numerical answer is required, an exact answer must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working must be shown.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1

Consider the function $f(x) = x^2 \log_e(2x)$.

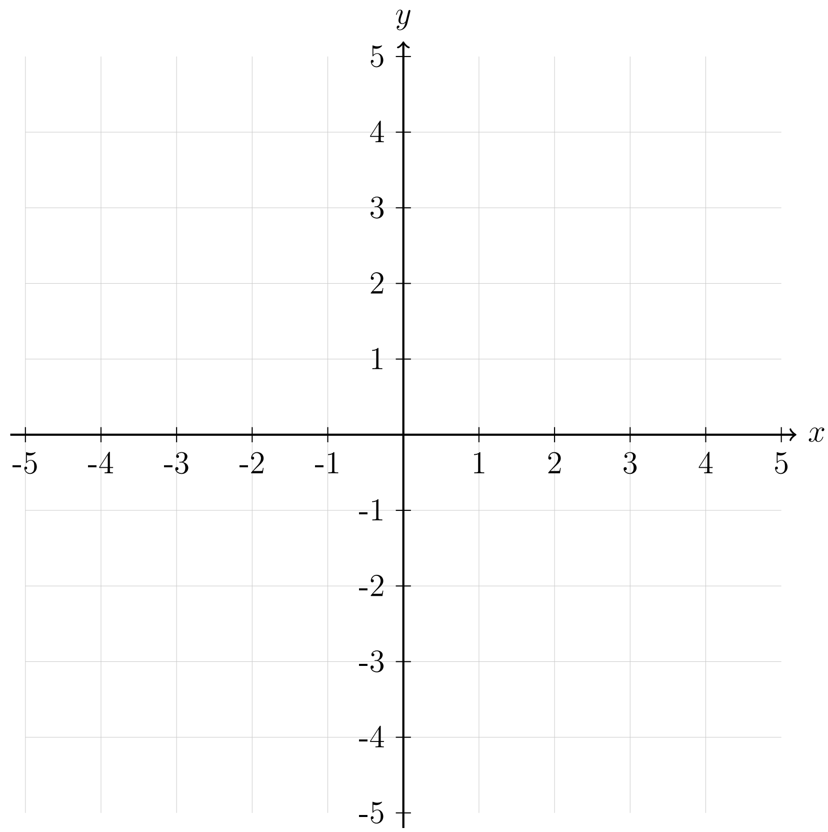
a) Find $f'(x)$. (1 mark)

b) Evaluate $\frac{d}{dx}[\log_e(x^2+1)]$ when $x = 2$. (2 marks)

Question 2

Consider the function $f(x) = \log_e(2x + 5)$.

a) Sketch the graph of $f(x)$. (3 marks)



b) Find the vertical asymptote, draw and label it on the graph. (1 mark)

c) Find the x- and y-intercepts and write their coordinates on the graph. (1 mark)

Question 3

Consider the functions

$$f : [-2, 2] \to \mathbb{R}, \quad f(x) = \sqrt{4 - x^2},$$

$$g : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad g(x) = \frac{1}{x}.$$

a) Find the range of the function f. (1 mark)

b) Find an expression for $(f \circ g)(x)$ and state its domain. (2 marks)

c) Given the codomain of f is $\mathbb{R}$, state the maximal domain of f for which the composition $(g \circ f)(x)$ exists. (1 mark)

Question 4

Find the general solution of the following equation: $\cos(2x + \frac{\pi}{6}) = -\frac{1}{2}$.

Give your answer in the form: $x = \ldots$, where $x \in \mathbb{R}$.

(3 marks)

Question 5

Consider the system of simultaneous equations:

$$\begin{aligned} 2x + (k+1)y &= 6 \\ kx + 6y &= m \end{aligned}$$

where k and m are real constants.

Find the integer value(s) of k and m for which the system has **no solution**. (4 marks)

Question 6

Let X be a binomial random variable where $X \sim \text{Bi}(4, \frac{1}{2})$.

a) Find the probability that exactly 2 successes occur. (1 mark)

b) Find the probability that at most 1 success occurs. (2 marks)

Question 7

A health study investigates the proportion of people in a small town who take vitamin supplements regularly. A random sample of people is surveyed, and 60% of them say they take supplements.

a) Let $X \sim \text{Bi}(n, p)$ represent the number of people who take supplements in a sample of size n, where the true proportion is $p = 0.6$. State the mean μ and standard deviation σ of X in terms of n. (2 marks)

b) Using a normal approximation, write the formula for a 95% confidence interval for the proportion p, given that the sample population is 24, and the sample proportion is $\hat{p} = 0.6$. (2 marks)

c) Suppose we want the standard deviation of the sample proportion to be no more than 0.05. Find the minimum sample size n required to achieve this. (2 marks)

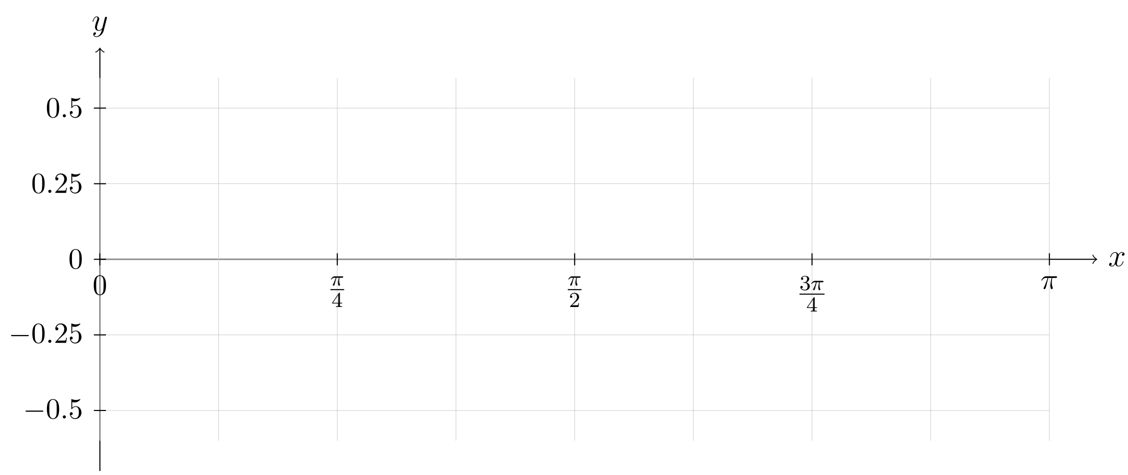
Question 8

Consider the function $f(x) = \sin(x)\cos(x)$ for $x \in [0, \pi]$.

a) Show that $f'(x) = \cos^2(x) - \sin^2(x)$. (2 marks)

b) Find the exact values of the coordinates of the stationary points in the interval $[0, \pi]$. (2 marks)

c) On the axes below, sketch the graph of $y = f(x) = \sin(x)\cos(x)$ for $x \in [0, \pi]$, labelling the stationary points with their exact coordinates. (2 marks)



d) Find the exact value of the area bounded by the curve $y = \sin(x)\cos(x)$ and the x-axis for $x \in [0, \pi]$. (2 marks)

Question 9

Solve the following equation for x.

$$2\log_5(x-1) = 1 + \log_5(x+1.8)$$

(4 marks)

Answer Key

Solution 1

Consider the function $y = x^2 \log_e(2x)$

Part (a) *(1 mark)*

Using the Product Rule:

$$f'(x) = x^2 \cdot \frac{d}{dx}[\log_e(2x)] + \log_e(2x) \cdot \frac{d}{dx}[x^2]$$

Step 1: Find $\frac{d}{dx}[\log_e(2x)]$

$$\frac{d}{dx}[\log_e(2x)] = \frac{1}{2x} \cdot 2$$

$$= \frac{1}{x}$$

Step 2: Find $\frac{d}{dx}[x^2] = 2x$
Step 3: Substitute into Product Rule

$$f'(x) = x^2 \cdot \frac{1}{x} + \log_e(2x) \cdot 2x$$

$$= x + 2x \log_e(2x)$$

Part (b) *(2 marks)*

Find $\frac{d}{dx}[\log_e(x^2 + 1)]$ when $x = 2$

Using the Chain Rule:

$$\frac{d}{dx}[\log_e(x^2 + 1)] = \frac{1}{x^2 + 1} \cdot \frac{d}{dx}[x^2 + 1] = \frac{1}{x^2 + 1} \cdot 2x$$

$$= \frac{2x}{x^2 + 1}$$

When $x = 2$:

$$= \frac{2(2)}{2^2 + 1} = \frac{4}{5}$$

Solution 2

a) The domain of $f(x) = \log_e(2x+5)$ requires the argument of the logarithm to be positive:

$$2x + 5 > 0 \implies x > -\frac{5}{2} = -2.5$$

The graph increases slowly and is continuous for $x > -2.5$. *(3 marks)*

b) The vertical asymptote is where the argument of the logarithm is zero:

$$2x + 5 = 0 \implies x = -2.5$$

(1 mark)

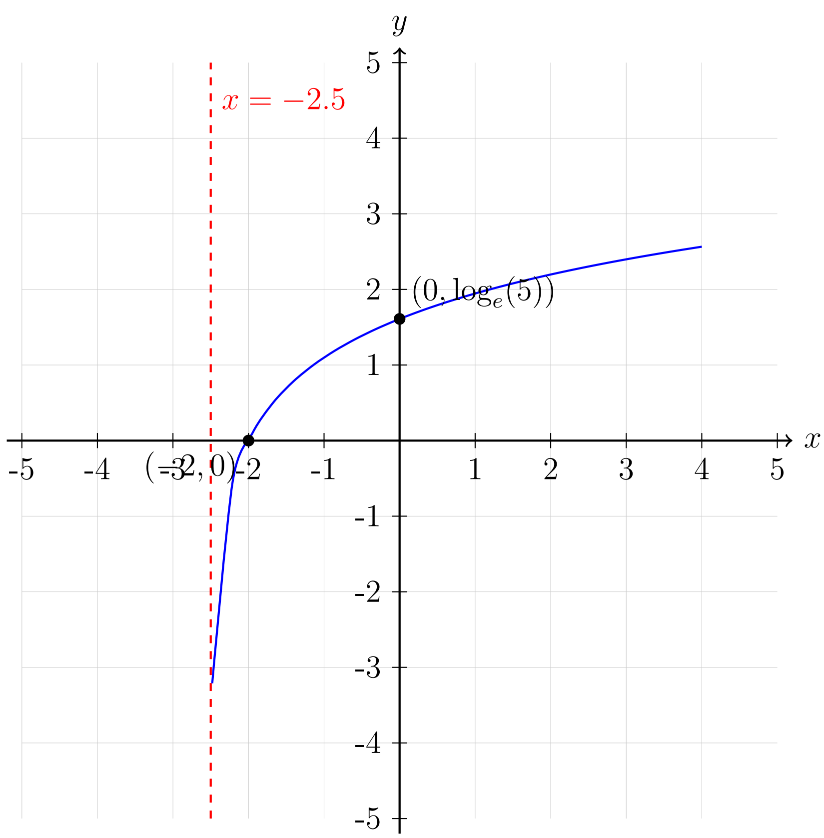
c) *Finding the y-intercept:* Set $x = 0$:

$$f(0) = \log_e(5)$$

Finding the x-intercept: Set $f(x) = 0$:

$$\log_e(2x+5) = 0 \implies 2x + 5 = 1 \implies 2x = -4 \implies x = -2$$

(1 mark)



Solution 3

a) Find the range of the function f. *(1 mark)*

We are given:

$$f(x) = \sqrt{4 - x^2}, \quad x \in [-2, 2]$$

Note: - The expression inside the square root, $4 - x^2$, is non-negative for $x \in [-2, 2]$. - The maximum occurs when $x = 0$: $f(0) = \sqrt{4} = 2$ - The minimum occurs when $x = \pm 2$: $f(\pm 2) = \sqrt{0} = 0$

Therefore, the range of f is

$$[0, 2]$$

b) Find an expression for $(f \circ g)(x)$ and state its domain. *(2 marks)*

We are finding:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \sqrt{4 - \left(\frac{1}{x}\right)^2} = \sqrt{4 - \frac{1}{x^2}}$$

This expression is only defined if:

$$4 - \frac{1}{x^2} \geq 0 \quad \Rightarrow \quad \frac{1}{x^2} \leq 4 \quad \Rightarrow \quad x^2 \geq \frac{1}{4}$$

Solving by factoring:

$$x^2 - \frac{1}{4} \geq 0 \quad \Rightarrow \quad \left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) \geq 0$$

Using a sign diagram or inequality rules, the solution is:

$$x \leq -\frac{1}{2} \quad \text{or} \quad x \geq \frac{1}{2}$$

Also, $x \neq 0$, since $g(x) = \frac{1}{x}$ is undefined at 0.

Domain:

$$(-\infty, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, \infty)$$

c) Given the codomain of f is $\mathbb{R}$, state the maximal domain of f for which the composition $(g \circ f)(x)$ exists. *(1 mark)*

We are finding:

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{\sqrt{4 - x^2}}$$

This is defined when: - $x \in [-2, 2]$ (from domain of f) - $f(x) \neq 0$

Since $f(x) = 0$ when $x = \pm 2$, we must exclude those points.

Maximal domain:

$$(-2, 2)$$

Solution 4

Solution: *3 marks*

We are given the equation:

$$\cos(2x + \frac{\pi}{6}) = -\frac{1}{2}$$

Step 1: Solve the auxiliary equation *(1 mark)*

Recall that:

$$\cos(\theta) = -\frac{1}{2} \quad \Rightarrow \quad \theta = \frac{2\pi}{3} + 2n\pi \quad \text{or} \quad \theta = \frac{4\pi}{3} + 2n\pi \quad \text{for } n \in \mathbb{Z}$$

Let $\theta = 2x + \frac{\pi}{6}$. So:

$$2x + \frac{\pi}{6} = \frac{2\pi}{3} + 2n\pi \quad \text{or} \quad 2x + \frac{\pi}{6} = \frac{4\pi}{3} + 2n\pi$$

Step 2: Solve for x *(1 mark)*

First solution:

$$2x + \frac{\pi}{6} = \frac{2\pi}{3} + 2n\pi$$

$$2x = \frac{2\pi}{3} - \frac{\pi}{6} + 2n\pi = \frac{4\pi}{6} - \frac{\pi}{6} + 2n\pi = \frac{3\pi}{6} + 2n\pi = \frac{\pi}{2} + 2n\pi$$

$$x = \frac{\pi}{4} + n\pi$$

Second solution:

$$2x + \frac{\pi}{6} = \frac{4\pi}{3} + 2n\pi$$

$$2x = \frac{4\pi}{3} - \frac{\pi}{6} + 2n\pi = \frac{8\pi}{6} - \frac{\pi}{6} + 2n\pi = \frac{7\pi}{6} + 2n\pi$$

$$x = \frac{7\pi}{12} + n\pi$$

Final Answer: *(1 mark)*

$$\boxed{x = \frac{\pi}{4} + n\pi \quad \text{or} \quad x = \frac{7\pi}{12} + n\pi, \quad \text{where } n \in \mathbb{Z}}$$

Solution 5

Step 1: Write down the system of equations:

$$\begin{cases} 2x + (k+1)y = 6 \\ kx + 6y = m \end{cases}$$

Step 2: Condition for no solution:
For no solution, the coefficients must satisfy:

$$\frac{\text{coefficient of } x \text{ in eq. 1}}{\text{coefficient of } x \text{ in eq. 2}} = \frac{\text{coefficient of } y \text{ in eq. 1}}{\text{coefficient of } y \text{ in eq. 2}} \neq \frac{\text{constant term in eq. 1}}{\text{constant term in eq. 2}}$$

Step 3: Write the ratio equations: *(1 mark)*

$$\frac{2}{k} = \frac{k+1}{6} \quad \text{and} \quad \frac{2}{k} \neq \frac{6}{m}$$

Step 4: Solve the equality of the coefficients: *(1 mark)*

$$\frac{2}{k} = \frac{k+1}{6}$$

Cross-multiplied:

$$12 = k(k+1)$$

$$k^2 + k - 12 = 0$$

Step 5: Solve the quadratic equation for k: *(1 mark)*

$$(k+4)(k-3) = 0)$$

So,

$$k = 3 \quad \text{or} \quad k = -4$$

Step 6: Write the inequality for no solution:

$$\frac{2}{k} \neq \frac{6}{m} \implies \frac{6}{m} \neq \frac{2}{k}$$

Rearranged:

$$m \neq \frac{6k}{2} = 3k$$

Step 7: Final answer: *(1 mark)*

$$\boxed{\begin{cases} k = 3 \text{ or } k = -4 \\ m \neq 9 \text{ and } k \neq -12 \end{cases}}$$

Solution 6

a) Use the binomial probability formula: *(1 mark)*

$$P(X=2)=\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=\binom{4}{2}\cdot\left(\frac{1}{2}\right)^4=6\cdot\frac{1}{16}=\frac{6}{16}=\frac{3}{8}$$

Answer: $P(X=2)=\frac{3}{8}$

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b) We want: *(2 marks)*

$$P(X\leq 1)=P(X=0)+P(X=1)$$

$$P(X=0)=\binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4=1\cdot 1\cdot\frac{1}{16}=\frac{1}{16}$$

$$P(X=1)=\binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3=4\cdot\frac{1}{2}\cdot\frac{1}{8}=4\cdot\frac{1}{16}=\frac{4}{16}=\frac{1}{4}$$

$$P(X\leq 1)=\frac{1}{16}+\frac{1}{4}=\frac{1}{16}+\frac{4}{16}=\frac{5}{16}$$

Answer: $P(X\leq 1)=\frac{5}{16}$

Solution 7

a) The mean and standard deviation of a binomial distribution $X \sim \text{Bi}(n, p)$ are: *(2 marks)*

$$\mu = np = 0.6n, \quad \sigma = \sqrt{np(1-p)} = \sqrt{n \cdot 0.6 \cdot 0.4} = \sqrt{0.24n}$$

b) Using a normal approximation, the standard deviation of the sample proportion is: *(2 marks)*

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.6 \cdot 0.4}{24}} = \sqrt{\frac{0.24}{24}} = \sqrt{0.01} = 0.1$$

The formula for a 95% confidence interval is:

$$\hat{p} \pm z \cdot \sigma_{\hat{p}} = 0.6 \pm 2 \cdot 0.1 = 0.6 \pm 0.2$$

$$(0.4,\ 0.8)$$

c) We want: *(2 marks)*

$$\sigma_{\hat{p}} = \sqrt{\frac{0.24}{n}} \leq 0.05$$

Square both sides:

$$\frac{0.24}{n} \leq 0.0025 \quad \Rightarrow \quad n \geq \frac{0.24}{0.0025} = \frac{24}{0.25} = 96$$

Therefore, a sample size of at least 96 is required.

Solution 8

Consider the function $f(x) = \sin(x)\cos(x)$ for $x \in [0, \pi]$.

a) Show that $f'(x) = \cos^2(x) - \sin^2(x)$. *(2 marks)*

Solution:

Using the product rule:

$$f(x) = \sin(x)\cos(x) \implies f'(x) = \sin'(x)\cos(x) + \sin(x)\cos'(x).$$

Since $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$, we have:

$$f'(x) = \cos(x)\cos(x) + \sin(x)(-\sin(x)) = \cos^2(x) - \sin^2(x).$$

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b) Find the exact values of the coordinates of the stationary points in the interval $[0, \pi]$. *(2 marks)*

Solution:

Stationary points occur where $f'(x) = 0$, i.e.

$$\cos^2(x) - \sin^2(x) = 0.$$

This is equivalent to

$$\cos^2(x) = \sin^2(x).$$

Divide both sides by $\cos^2(x)$ (where defined):

$$1 = \tan^2(x) \implies \tan(x) = \pm 1.$$

Within $[0, \pi]$, the solutions are

$$x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}.$$

Now substitute into $f(x) = \sin(x)\cos(x)$:

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$$

$$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$$

Therefore, the stationary points are $\left(\frac{\pi}{4}, \frac{1}{2}\right)$ and $\left(\frac{3\pi}{4}, -\frac{1}{2}\right)$.

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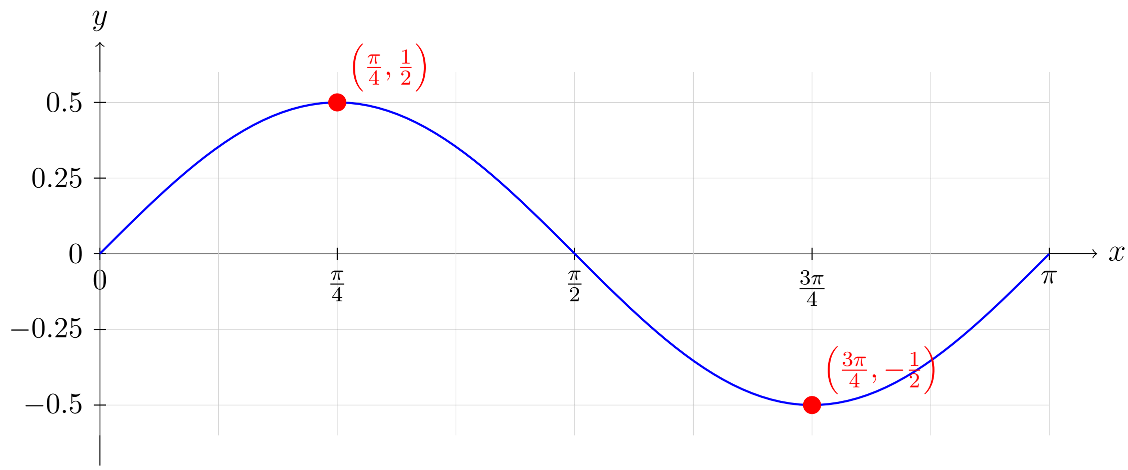
c) On the axes below, sketch the graph of $y = f(x) = \sin(x)\cos(x)$ for $x \in [0, \pi]$, labelling the stationary points with their exact coordinates. *(2 marks)*



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d) Find the exact value of the area bounded by the curve $y = \sin(x)\cos(x)$ and the x-axis for $x \in [0, \pi]$. *(2 marks)*

Solution:

The curve $y = \sin(x)\cos(x)$ crosses the x-axis at $x = 0, \frac{\pi}{2}, \pi$. Note that on $[0, \frac{\pi}{2}]$, $y \geq 0$ and on $[\frac{\pi}{2}, \pi]$, $y \leq 0$.

The area bounded by the curve and the x-axis is given by

$$\text{Area} = \int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx - \int_{\frac{\pi}{2}}^{\pi} \sin(x)\cos(x)\,dx,$$

because the integral on $[\frac{\pi}{2}, \pi]$ is negative and subtracting it adds its absolute value.

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Evaluate the first integral:

Let

$$u = \sin(x) \implies du = \cos(x)dx.$$

When $x = 0, u = \sin(0) = 0$, and when $x = \frac{\pi}{2}, u = \sin\left(\frac{\pi}{2}\right) = 1$.
Thus,

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

—

Evaluate the second integral:
Using the same substitution $u = \sin(x)$, when $x = \frac{\pi}{2}, u = 1$, and when $x = \pi, u = \sin(\pi) = 0$.
Therefore,

$$\int_{\frac{\pi}{2}}^{\pi} \sin(x)\cos(x)\,dx = \int_1^0 u\,du = -\int_0^1 u\,du = -\left[\frac{u^2}{2}\right]_0^1 = -\frac{1}{2}.$$

—

Calculate total area:

$$\text{Area} = \frac{1}{2} - (-\frac{1}{2}) = \frac{1}{2} + \frac{1}{2} = 1.$$

Final answer:

$$1.$$

Solution 9

We are given: *(4 marks)*

$$2\log_5(x-1) = 1 + \log_5(x+1.8)$$

Step 1: Apply the power rule to bring the coefficient inside the logarithm:

$$\log_5((x-1)^2) = 1 + \log_5(x+1.8)$$

Step 2: Convert the constant term to a logarithm:

$$1 = \log_5(5)$$

So:

$$\log_5((x-1)^2) = \log_5(5) + \log_5(x+1.8)$$

Step 3: Combine the right-hand side logs:

$$\log_5((x-1)^2) = \log_5(5(x+1.8)) = \log_5(5x+9)$$

Step 4: Since the logarithms are equal (same base), equate the arguments:

$$(x-1)^2 = 5x+9$$

Step 5: Expand and rearrange into standard quadratic form:

$$x^2 - 2x + 1 = 5x + 9$$

$$x^2 - 2x + 1 - 5x - 9 = 0$$

$$x^2 - 7x - 8 = 0$$

Step 6: Factorise the quadratic. Find two numbers that multiply to -8 and add to -7:

$$-8 = (-8) \times 1, \quad (-8) + 1 = -7$$

So:

$$x^2 - 7x - 8 = (x-8)(x+1) = 0$$

Step 7: Set each factor to zero:

$$x - 8 = 0 \quad \Rightarrow \quad x = 8$$

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$

Step 8: Check domain restrictions:

$$x - 1 > 0 \Rightarrow x > 1, \quad x + 1.8 > 0 \Rightarrow x > -1.8$$

Only $x = 8$ satisfies both.

Final answer: $x = 8$